FEATURE

11 Problems

The HCMR welcomes submissions of original problems in any fields of mathematics, as well as solutions to previously proposed problems. Proposers should direct problems to hcmr-problems@hcs.harvard.edu or to the address on the inside front cover. A complete solution or a detailed sketch of the solution should be included, if known. Solutions to previous problems should be directed to hcmr-solutions@hcs.harvard.edu or to the address on the inside front cover. Solutions should include the problem reference number, as well as the solver's name, contact information, and affiliated institution. Additional information, such as generalizations or relevant bibliographical references, is also welcome. Correct solutions will be acknowledged in future issues, and the most outstanding solutions received will be published. To be considered for publication, solutions to the problems below should be postmarked no later than *March 1, 2008*. An asterisk beside a problem or part of a problem indicates that no solution is currently available.

F07 – 1. Consider $\triangle ABC$ an arbitrary triangle and P a point in its plane. Let D, E, and F be three points on the lines through P perpendicular to the lines $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Prove that if $\triangle DEF$ is equilateral and if P lies on the Euler line of $\triangle ABC$, then the center of $\triangle DEF$ also lies on the Euler line of $\triangle ABC$.

Proposed by Cosmin Pohoata (Bucharest, Romania) and Darij Grinberg (Germany).

F07 – 2. Professor Perplex has rounded up his $n > 0$ hat-game seminar students and made the following ominous announcement:

> "I have assigned each of you a hat according to a uniform probability distribution, which I will put on your head after allowing you time to discuss a strategy. Hats come in $h > 0$ different colors, but some colors might be reused and others might not be used at all. Each student will be given a list of the h colors. Nobody will be able to see his or her own hat, but everyone will have the opportunity to observe all the other hats. Then, you will all be instructed to simultaneously write down one of the colors. If any student correctly identifies the color of his or her own hat, then there will be no final exam this semester. Otherwise, I will assign a week-long haberdashery final."

What is the probability that the students have to take a final, assuming best play?

Proposed by John Hawksley (Massachusetts Institute of Technology '08) and Scott Kominers '09.

F07 – 3. Find all integer monic polynomials $f(x)$ such that

(i) $f(x) = f(1 - x)$ and

(ii) all complex zeros of f lie in the disk $|z| < \sqrt[5]{2}$.

Proposed by Vesselin Dimitrov '09.

F07 – 4. Let $a, b \geq 0$ be two nonnegative numbers. Find the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k+b+\sqrt{n^2+kn+a}}.$$

Proposed by Ovidiu Furdui (University of Toledo).

F07 – 5. For $i = 1, \dots, n$, let $f_i : (\mathbb{Z}/m\mathbb{Z} \cup \{\star\})^n \to (\mathbb{Z}/m\mathbb{Z} \cup \{\star\})^n$ be given by

$$f_i(\vec{x}) = \begin{cases} (\star, x_2+1, x_3, \dots, x_n) & i = 1 \text{ and } x_1 = 1, \\ (x_1, \dots, x_{i-2}, x_{i-1}+1, \star, x_{i+1}+1, x_{i+2}, \dots, x_n) & 1 < i < n \text{ and } x_i = 1, \\ (x_1, \dots, x_{n-2}, x_{n-1}+1, \star) & i = n \text{ and } x_n = 1, \\ (x_1, \dots, x_n) & \text{otherwise,} \end{cases}$$

where $\star + 1 = \star$ and $\vec{x} = (x_1, \dots, x_n)$. Find necessary and sufficient conditions on $(x_1, \dots, x_n) \in (\mathbb{Z}/m\mathbb{Z})^n$ such that there exists a sequence $\{i_k\}_{k=1}^n$ for which

$$f_{i_n}(\cdots(f_{i_1}(\vec{x}))) = (\star, \dots, \star).$$

Proposed by Paul Kominers (Walt Whitman HS '08), Scott Kominers '09, and Zachary Abel '10.

The following two problems from the Spring 2007 issue received a total of one submission: a correction for S07 – 4 by Alon Amit (Google), for which we are most grateful. Since these problems defied solution, we are re-releasing them for one more issue. Their solutions will appear in Spring 2008.

S07 – 3. The incircle Ω_{ABC} of a triangle ABC is tangent to BC, CA, AB at P, Q, R respectively. Rays PQ and BA intersect at M, rays PR and CA intersect at N, and the incircle Ω_{MNP} of triangle MNP is tangent to MN and NP at X and Y respectively. Given that X, Y and B are collinear, prove:

(a) Circles Ω_{ABC} and Ω_{MNP} are congruent, and

(b) these circles intersect each other in 60° arcs.

Proposed by Zachary Abel '10.

S07 – 4 (Corrected)**.** For a prime p, let $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ denote the localization of the integral domain $\mathbb{Z}$ at the prime ideal (p); that is, the subring of $\mathbb{Q}$ consisting of the rational numbers with denominators prime to p. The canonical homomorphism $\mathbb{Z} \to \mathbb{F}_p$ induces a canonical homomorphism $\phi_p : \mathbb{Z}_{(p)} \to \mathbb{F}_p$, the reduction modulo p homomorphism with kernel the maximal ideal $p\mathbb{Z}_{(p)}$ of the local ring $\mathbb{Z}_{(p)}$. For example, $\phi_5(1/2) = 3 \in \mathbb{F}_5$.

Let V be the set of primes p for which $\{\frac{3^n-1}{2^n-1} \mid n \in \mathbb{N}\} \subset \mathbb{Z}_{(p)}$.

(a) Characterize the set V.

(b) Let P be the set of primes, and define the set $W \subset P$ of **Wieferich primes** to be the set of primes p such that $p^2 \mid 2^{p-1} - 1$. It has been conjectured that, as x tends to infinity, the size of $\{p \in W \mid p \leq x\}$ is $O(\log \log x)$.

Show that V and $P \setminus V$ are both infinite sets, assuming the above conjecture for the former.

(c) Show that, for every $p \in V$, the map $\mathbb{N} \to \mathbb{F}_p$ given by $n \mapsto \phi_p\big((3^n - 1)/(2^n - 1)\big)$ is periodic.

For example, $5 \in V$, and the corresponding map $\mathbb{N} \to \mathbb{F}_5$ is $2, 1, 3, 2, 2, 1, 3, 2, 2, 1, 3, 2, \dots$.

Proposed by Vesselin Dimitrov '09.

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12 Solutions

How to Chop a Hyperbox

S07 – 1. How many hyperplane cuts are necessary to divide a $3 \times 5 \times 7 \times 9 \times 11$ rectangular solid into $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$ distinct $1 \times 1 \times 1 \times 1 \times 1$ hypercubes, if previously separated pieces can be rearranged between cuts?

Proposed by Joel Lewis '07.

Solution by Alon Amit (Google). Each cut can, at best, double the number of solid pieces, so an obvious lower bound is $\lceil \log_2(V) \rceil$, where V is the volume of the rectangular solid (henceforth "the box"). However, edges of odd length cannot be efficiently halved, so we are led to the following:

Proposition 3. *Let V be a box of dimensions $(a_1, a_2, \ldots a_n)$. The box can be cut into unit cubes using $L(V)$ cuts, and no fewer, where $L(V)$ is*

$$L(V) = L(a_1, \ldots, a_n) = \sum_{i=1}^{n} \lceil \log_2(a_i) \rceil.$$

Proof. For a fixed dimension $n \geq 1$, we prove this by induction on the value of L. We have $L = 0$ if and only if the box is a unit cube to begin with, so the claim holds in this case. We now assume the claim holds for all boxes with L-value less than L, and prove it for a box V with $L(V) = L \geq 1$. We need to show that V can be fully chopped with the advertised number of cuts and that any chopping procedure requires at least that many cuts.

Since $L \geq 1$, V has at least one edge whose length a_i is greater than 1. We cut the box across this edge, as close to the middle as possible. Namely, letting $b_i = \lceil a_i/2 \rceil$, we cut V into two boxes $V_1 = (a_1, \ldots, b_i, \ldots a_n)$ and $V_2 = (a_1, \ldots, a_i - b_i, \ldots, a_n)$.

Note that $L(V_1) = L(V) - 1$ and $L(V_2) \leq L(V_1)$. By the inductive hypothesis, V_1 can be chopped with $L(V_1)$ cuts. Moreover, V_2 can be chopped with that same number of cuts (or less), and these can be performed simultaneously with those of V_1: simply rearrange the pieces of V_2 that need to be cut at each stage along the same hyperplane used for cutting V_1. We are thus able to fully chop both V_1 and V_2 with $L(V) - 1$ cuts. Together with the initial cut, then, we have cut V in $L(V)$ cuts.

Furthermore, *any* procedure for fully chopping V must start with a single cut creating two pieces, each identical to V in all dimensions save one, and the larger of which is at least half as large as V. It follows that the first cut creates a piece W with $L(W) \geq L(V) - 1$. By the inductive hypothesis, W cannot be fully chopped with less than $L(V) - 1$ cuts, so the original box V requires at least $L(V)$ cuts. □

It is interesting to note that the proof shows a bit more than claimed: to optimally chop a box, all one needs to do is choose a non-trivial edge in each piece currently on hand and simultaneously cut them all near or at their middle. No further cleverness is required in choosing the sides or the cut locations.

For the box in the original problem, the number of cuts required is

$$L(3, 5, 7, 9, 11) = 2 + 3 + 3 + 4 + 4 = 16. \qquad \square$$

Also solved by Sergey Ioffe (Google), and the proposer.

πs in Odd Places

S07 – 2. Suppose $f:[0,1]\to\mathbb{R}$ is an integrable function such that $f(x)y+f(y)x\le x^2+y^2$. Show that $\int_0^1 f(x)\,dx\le\frac{\pi}{4}$. (One example of such a function is $f(x)=x$.)

Proposed by Scott Kominers '09.

Solution by Garret Dan Vo (Montana State University, Bozeman '10). Integrating the given inequality $x\cdot f(y)+y\cdot f(x)\le x^2+y^2$ over the unit square $(x,y)\in[0,1]\times[0,1]$ gives the following stronger bound (by Fubini's Theorem):

$$\begin{aligned}\int_0^1 f(x)\,dx=\frac{1}{2}\int_0^1 f(y)\,dy+\frac{1}{2}\int_0^1 f(x)\,dx&=\int_0^1\int_0^1\big(x\cdot f(y)+y\cdot f(x)\big)\,dx\,dy\\&\le\int_0^1\int_0^1(x^2+y^2)\,dx\,dy=\frac{2}{3}<\frac{\pi}{4}.\qquad\square\end{aligned}$$

Solution by Noam D. Elkies (Harvard University). Setting $x=y$ in the original constraint gives

$$x\cdot f(x)\le x^2,$$

whence $f(x)\le x$ for $x>0$. Thus, we have for any such f that

$$\int_0^1 f(x)\,dx\le\int_0^1 x\,dx=\frac{1}{2}<\frac{\pi}{4}.\qquad\square$$

Solution by the proposer. Making the substitution $(x,y)=(\cos t,\sin t)$ for $t\in[0,\frac{\pi}{2}]$ reduces the constraint to $\sin t\cdot f(\cos t)+\cos t\cdot f(\sin t)\le 1$. Integrating this gives the desired bound:

$$\int_0^1 f(x)\,dx=\frac{1}{2}\int_0^{\frac{\pi}{2}}\big(\sin t\cdot f(\cos t)+\cos t\cdot f(\sin t)\big)\,dt\le\frac{1}{2}\int_0^{\frac{\pi}{2}}dt=\frac{\pi}{4}.\qquad\square$$

Also solved by Sherry Gong '11, John Hawksley (Massachusetts Institute of Technology '08), Sergey Ioffe (Google), Daniel Litt '10, Greg Price '06–'07, The Northwestern University Math Problem Solving Group, Manuel Silva (New University of Lisbon), and Arnav Tripathy '11.

Yet Another Mean Inequality

S07 – 3. (a) Prove that for distinct positive real numbers a and b, the following inequality holds:

$$\frac{a+b}{2}\ge\frac{a^{\frac{a}{a-b}}\,b^{\frac{b}{b-a}}}{e}\ge\frac{a-b}{\ln a-\ln b}.$$

(b*) Show that both inequalities are strict.

Proposed by Shrenik Shah '09.

Solution to parts (a) and (b) by Greg Price '06–'07. Consider the function $f:(-1,1)\to\mathbb{R}$, $f(x)=\frac{1}{2x}\ln\frac{1+x}{1-x}$, with $f(0)=1$. Observe that f is analytic and that it is given by the power series

$$f(x)=\frac{1}{2x}(\ln(1+x)-\ln(1-x))=1+\frac{x^2}{3}+\frac{x^4}{5}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n}}{2n+1}$$

on the entire interval $(-1, 1)$.

We will need three facts about f. First, $f(x) \geq 1$, with equality only at $x = 0$; this follows immediately from the power series. Second, $f(x) \leq (1 - x^2)^{-1/2}$, with equality only at $x = 0$; this follows again from the power series, as the right-hand side is given by

$$1 + \frac{1}{2}x^2 + \frac{1}{2} \cdot \frac{3}{4}x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2} x^{2n}$$

on the whole interval $(-1, 1)$, and for all $n > 0$ we have $\frac{1}{2n+1} < \frac{(2n)!}{(2^n n!)^2}$ by a simple induction. Third, $f'(x) = \frac{1}{x(1-x^2)} - \frac{f(x)}{x}$; this follows from an elementary differentiation of $f(x) = \frac{1}{2x} \ln \frac{1+x}{1-x}$.

Now, given distinct positive reals a, b, we wish to prove

$$\frac{a+b}{2} > \frac{a^{\frac{a}{a-b}} b^{\frac{b}{b-a}}}{e} > \frac{a-b}{\ln a - \ln b}.$$

Let $x = \frac{a-b}{a+b}$; dividing through by $\frac{a+b}{2}$, the desired inequalities become

$$1 > (1 - x^2)^{1/2} e^{f(x)-1} > \frac{1}{f(x)},$$

which we now wish to prove for nonzero x in $(-1, 1)$. Since the three expressions are always positive, we may pass to their logarithms; since they are invariant under $x \mapsto -x$ we may consider only $x > 0$; since they are equal at $x = 0$, it will suffice to show that their respective logarithmic derivatives obey the same inequalities. So we wish to show for $x \in (0, 1)$ that $0 > -\frac{x}{1-x^2} + f'(x) > -\frac{f'(x)}{f(x)}$, or equivalently, multiplying by x, subtracting from 1, and employing our computation of the derivative f', that

$$1 < f(x) < \frac{1}{f(x)(1 - x^2)}.$$

But the left inequality follows from our first fact above, and the right from our second. We have proved the desired inequalities. □

Editor's note. The proposer noted that the inequality in the problem was obtained by taking the limit of the AM-GM-HM inequality for an arithmetic sequence of n terms from a to b as n goes to infinity. Paolo Perfetti (Università degli Studi Di Roma "Tor Vergata") pointed out that part (a) is a direct consequence of problem E3142 in *The American Mathematical Monthly* **95**#3, proposed by Zhang Zaiming. The solution provided there, by Ricardo Perez Marco, additionally proves part (b).

Also solved by Vishal Lama (Southern Utah University) and Paolo Perfetti (Università degli studi di Roma "Tor Vergata"). The proposer solved part (a), only. Partial solutions to part (a) were provided by Avery Carr (University of Memphis) and The Northwestern University Math Problem Solving Group.